Measurements and Units

You often solve problems in science by doing arithmetic on *measurements*. A measurement is not just a number; it is more than a number. A measurement has two parts: a number and a name, called the measurement's *unit*.

Correct arithmetic on measurements *requires* that you include the units in the arithmetic. You do this by treating the unit name just like a numeric *factor*. Arithmetic on measurements then becomes a simple extension of arithmetic on numbers.

Measurements

The data of science consist of measurements. A measurement is written or spoken in two parts: a number and a name; the name part is called the measurement's *unit*. Examples of measurements are

3 feet, 0.5 liter, 2200 grams.

"Feet", "liters", and "grams" are units.

In a measurement the number part without the unit part is meaningless. You can see this by noticing that both "1 mile" and "5280 ft" are the same measurement. Since both "1 mile" and "5280 ft" describe the same measurement you can write the equation

1 mile = 5280 ft

but of course 1 is not equal to 5280. If you don't treat the unit as something equally important as the number you will be led astray.

The Arithmetic of Measurements

Factors are numbers that multiply together. The factors of a particular number are numbers that can multiply together to make this number. For example, 2, 3, 4, and 6 are factors of 12. What's new about measurements is that *units are factors of measurements*.

A unit word is a factor of the measurement in which it appears. For example, the factors of the measurement

12 in (i.e., 12 inches)

are

2, 3, 4, 6, and "in". [1]

In "12 in" you are multiplying the number "12" and the unit "in". If this seems weird it's because a measurement is more than just a number, and *it's not correct to think of a measurement as just a number*.

[1] Just ignore whether a unit word is written singular or plural; from now on we'll favor singular abbreviations for consistency.

Thinking of a unit as a factor of a measurement is all we need in order to understand the arithmetic of measurements. Here are the rules.

Addition Rule. In order to add or subtract two measurements, their units must be the same, i.e., "you can't add apples and oranges".[2]

Multiplication Rule. Here is one way you can multiply two numbers: just combine the factors of the two numbers. For example (there are may ways you can do this),

$$
\begin{aligned}
&12 \cdot 6 \\
&= (4 \cdot 3) \cdot (3 \cdot 2) \\
&= 4 \cdot 3 \cdot 3 \cdot 2 \\
&= 4 \cdot 9 \cdot 2 \\
&= 4 \cdot 18 \\
&= 72
\end{aligned}
$$

The multiplication rule: When multiplying and dividing measurements, the units must be treated as factors in the computation. That is, units can be multiplied and canceled. You will now see two examples.

Example of the Multiplication Rule: Area and Volume

Length, area, and volume are measurements that illustrate the multiplication rule. Here is an example. The length and width of a rectangle are measurements. The rule for computing the area of a rectangle is length · width, that is, the product of two measurements. (Area is also a measurement.)

Say the length of a rectangle is 6 cm and the width is 5 cm. Here is the area computation.

$$\text{area} = \text{length} \cdot \text{width} = 6\ \text{cm} \cdot 5\ \text{cm} = 6 \cdot 5 \cdot \text{cm} \cdot \text{cm} = 30\ \text{cm}^2$$

Here you see that "cm^2" is not just shorthand for saying "square centimeters"; it is actually the result of treating units as factors and multiplying cm by cm. *This is not a coincidence; it is how these things work.* That is, "cm^2" is a unit of area. *A unit of area is a product of two units of length.* Examples are cm^2 and in^2; even cm · mile would be a valid area unit.

The same idea holds for volume units. *A unit of volume is a product of three units of length.* Examples are m^3, cm^3, and in^3. (Here is a curious real-world example: acre · ft is used to measure water volume in agricultural irrigation; acre is an area unit.)

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[2] There actually is a proof that you can add apples and apples; use the distributive rule.

$$5\ \text{apple} + 6\ \text{apple} = (5 + 6)\ \text{apple} = (11)\ \text{apple} = 11\ \text{apple}.$$

But you can't use the distributive rule on

$$5\ \text{apple} + 6\ \text{orange}\ .$$

Multiplication of Fractions

When you multiply two fractions to make a product,

$$\frac{a}{b} \bullet \frac{c}{d}$$

the numerator of the product is the product of the numerators and the denominator of the product is the product of the denominators.

$$\frac{a}{b} \bullet \frac{c}{d} = \frac{a \bullet c}{b \bullet d}$$

(Note that multiplying fractions is easier than adding them.) Factors in the numerator and denominator of a fraction can be *canceled*:

$$\frac{2}{7} \bullet \frac{7}{13} = \frac{2 \bullet 7}{7 \bullet 13} = \frac{2}{13}$$

One more useful fact: for any number a,

$$a = \frac{a}{1}$$

Example of the Multiplication Rule: Unit Conversion

Unit conversion is the process of changing a measurement with one unit into *the same* measurement with another unit. It is a simple application of arithmetic on measurements. *It will free you from memorizing "multiply by" and "divide by" rules.*

Let's see how to find the number of inches in 3.6 yards. We are looking for the number that goes in the place of "?" in this equation:

3.6 yd = ? in

Here is the procedure. We start with this identity. (We shall refer to this as the *starting identity*.)

3.6 yd = 3.6 yd

We will leave the left side alone and change the right side so the "yd" goes away and "in" appears instead.

The key principle we use is that multiplying any measurement by 1 does not change it. *We are going to find ways of writing 1 that change the unit and the number on the right side but don't change the measurement; thus the equation will remain valid.*

Our strategy is based on the fact that we know how to get from yards to feet and then from feet to inches. Let's first go from yards to feet.

We know

$$1 \text{ yd} = 3 \text{ ft}$$

Since this is an equation it remains valid if we multiply or divide both sides by the same thing. If we divide both sides by 3 ft we get

$$\frac{1 \text{ yd}}{3 \text{ ft}} = 1$$

If we divide both sides by 1 yd we get

$$\frac{3 \text{ ft}}{1 \text{ yd}} = 1$$

This second one is our key because when we multiply the right side of the starting identity by this form of 1 the "yd" cancels out leaving "ft". Here is the right side of the starting identity.

$$3.6 \text{ yd} \bullet \left(\frac{3 \text{ ft}}{1 \text{ yd}}\right) = \left(\frac{3.6 \text{ yd}}{1}\right) \bullet \left(\frac{3 \text{ ft}}{1 \text{ yd}}\right) = \left(\frac{3.6 \text{ yd} \bullet 3 \text{ ft}}{1 \text{ yd}}\right) = \frac{3.6 \bullet 3 \text{ ft}}{1}$$

In order to show how the conversion can be done all in one line, we're not going to do the arithmetic yet until we get rid of the "ft" and end up with "in". We know

$$1 \text{ ft} = 12 \text{ in}$$

so we do the two usual divisions to get

$$\frac{1 \text{ ft}}{12 \text{ in}} = 1 \text{ and } \frac{12 \text{ in}}{1 \text{ ft}} = 1$$

The right-hand one does the trick because when we multiply the right side of the identity (as we just modified it) by this form of 1 the "ft" cancels out. Here is how you modify the starting identity on one line.

$$3.6 \text{ yd} = 3.6 \text{ yd} \bullet \left(\frac{3 \text{ ft}}{1 \text{ yd}}\right) \bullet \left(\frac{12 \text{ in}}{1 \text{ ft}}\right) = 3.6 \bullet 3 \bullet 12 \text{ in} = 129.6 \text{ in}$$

> **The trick, then, is to multiply the right side of the starting identity by fractions that equal 1 in order to cancel out the units you want to eliminate, and to introduce the units you want to end up with. This might take several steps, but can usually be done on one line. Once you become familiar with the method you can do unit conversions on the fly.**

Thinking of arithmetic on measurements as a simple extension of arithmetic on numbers is a more powerful technique than is generally appreciated, for two reasons.

- It can be used in a wider class of problems than conversion of units in measurements. For example, it greatly simplifies rate problems.
- You will learn to use it to *guide* your setup of many problems.